Supplementary Material: Swapping and Purification Scheme Optimization for Entanglement Distribution in Quantum Networks - Jiyao Liu, Xinwen Zhang, Xinliang Wei, Xuanzhang Liu, Yuzhou Chen, Hongchang Gao, and Yu Wang

In this supplementary material, we provide the fidelity formulas of swapping and purification in noisy Werner system, and formally prove Lemmas 2 and Theorem 3.

Swapping and Purification in Noisy Werner System

The fidelity of the output pair of a swapping functions S_w on the two input entangled pairs with f_1 and f_2 fidelity in noisy Werner systems can be modeled as follows [14, 23]

$$S^n_w(f_1, f_2) = \frac{1}{4} + \frac{3\alpha_1\alpha_2}{4}\frac{4\eta^2-1}{3}\frac{4f_1-1}{3}\frac{4f_2-1}{3}. \tag{40}$$

Here, α_1, α_2, η represent the fidelity of 1-qubit, 2-qubit, and BSM gates used for noisy Werner systems.

Define $\rho = \alpha_1 \times \alpha_2$ and $\phi_i = (1 - f_i)/3$, then the success probabilities of purification are given by [14, 15, 44, 47]

$$\Pr\{P^n_w\} = \rho^2[\eta^2 + (1-\eta)^2](f_1f_2 + f_1\phi_2 + \phi_1 f_2 + 5\phi_1\phi_2) + 2\rho^2\eta(1-\eta)(2f_1\phi_2 + 2f_2\phi_1 + 4\phi_1\phi_2) + \frac{1-\rho^2}{2}. \tag{41}$$

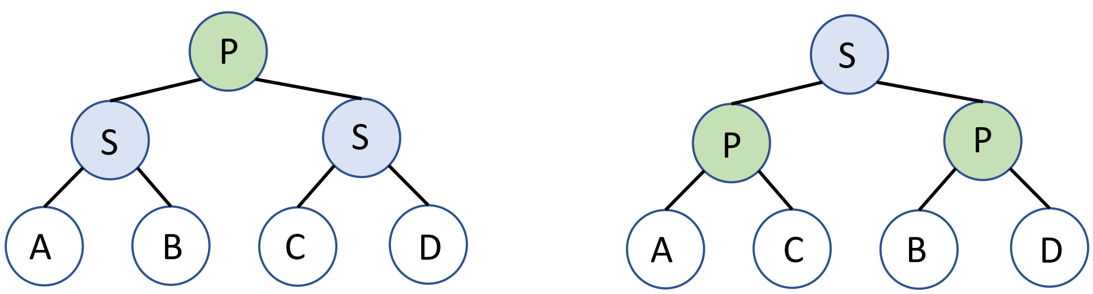
We can see that they are related to the fidelity of input entanglements. The fidelity of output entanglement of purification is defined as [14, 15, 44, 47]

$$P^n_w(f_1, f_2) = \frac{A(f_1, f_2, \rho, \eta)\rho^2}{\Pr\{P_w\}}, \tag{42}$$

where $A(f_1, f_2, \rho, \eta) = [\eta^2 + (1-\eta^2)](f_1f_2 + \phi_1\phi_2) + 2\eta(1-\eta)(f_1\phi_2 + f_2\phi_1) + \frac{1-\rho^2}{8\rho^2}$.

Proof of Lemma 2

We prove Lemma 2 by showing that a transformation from PRR to SRR (as shown in Fig. 1) can increase its fidelity and reduce cost at the same time. Here we only prove for a single step, where the node P is pushed one level down in the PRR. However, such steps can be repeated until the PRR changes to an SRR. Suppose the fidelity and expected cost of subtrees A, B, C, and D (in Fig 1) are f_1, f_2, f_3, f_4 and c_1, c_2, c_3, c_4, respectively. We now prove the fidelity part first, and then the cost part.



(a) before transformation (b) after transformation

Fig. 1: Transformation from PRR to SRR.

Fidelity Part of Lemma 2

We first prove the transformation increases the fidelity, i.e.,

$$S_p(P_p(f_1, f_3), P_p(f_2, f_4)) > P_p(S_p(f_1, f_2), S_p(f_3, f_4)). \tag{43}$$

Be aware that the two 'S' subtrees in Fig. 1(a) should be the same. If not, either one of them is not optimal or both are optimal. For both cases, we can simply adopt the optimal subtree to the other, so that both subtrees are the same. However, we still keep their fidelity different in this proof as it is a more general case, which might be useful for future analysis.

Let $s_{ij} = f_if_j + (1-f_i)(1-f_j) = f_if_j + \bar{f}_i\bar{f}_j$, then according to the definitions of S_p and P_p, Equ. (43) is

$$\frac{f_1f_2f_3f_4 + \bar{f}_1\bar{f}_2\bar{f}_3\bar{f}_4}{s_{13}s_{24}} > \frac{s_{12}s_{34}}{s_{12}s_{34} + (1-s_{12})(1-s_{34})}.$$

After a few simplifications, it becomes

$$(\Pi f_i)(1-s_{12})(1-s_{34}) + (\Pi \bar{f}_i)(1-s_{12})(1-s_{34}) > s_{12}s_{34}f_1f_3\bar{f}_2\bar{f}_4 + s_{12}s_{34}f_2f_4\bar{f}_1\bar{f}_3.$$

There are still more than 20 terms in this inequality after further expansion. Let L_1, L_2, R_1 and R_2 be the left and right terms of this inequality. We need to prove $L_1 + L_2 - R_1 - R_2 > 0$.

We can further simplify these four terms to get

$$L_1 = \Pi f_i - (\Pi f_i)f_3f_4 - (\Pi f_i)\bar{f}_3\bar{f}_4 - (\Pi f_i)f_1f_2 + (\Pi f_i)^2 + (\Pi f_i)f_1f_2\bar{f}_3\bar{f}_4 - (\Pi f_i)\bar{f}_1\bar{f}_2 + (\Pi f_i)\bar{f}_1\bar{f}_2f_3f_4 + (\Pi f_i)(\Pi \bar{f}_i).$$

$$L_2 = \Pi \bar{f}_i - (\Pi \bar{f}_i)f_3f_4 - (\Pi \bar{f}_i)(\bar{f}_3\bar{f}_4 - (\Pi \bar{f}_i)f_1f_2 + (\Pi f_i)(\Pi \bar{f}_i) + (\Pi \bar{f}_i)f_1f_2\bar{f}_3\bar{f}_4 - (\Pi \bar{f}_i)\bar{f}_1\bar{f}_2$$
$$+ (\Pi(1-f_i))(1-f_1)(1-f_2)f_3f_4 + \Pi(1-f_i)^2.$$

$$R_1 = (\Pi f_i)f_1f_3\bar{f}_2\bar{f}_4 + f_1^2f_2f_3\bar{f}_2\bar{f}_3\bar{f}_4{}^2 + f_1f_3^2f_4\bar{f}_1\bar{f}_2{}^2\bar{f}_4 + (\Pi \bar{f}_i)f_1f_3\bar{f}_2\bar{f}_4.$$

$$R_2 = (\Pi f_i)f_2f_4\bar{f}_1\bar{f}_3 + f_1f_2^2f_4\bar{f}_1\bar{f}_3{}^2\bar{f}_4 + f_2f_3f_4^2\bar{f}_1{}^2\bar{f}_2\bar{f}_3 + (\Pi \bar{f}_i)f_2f_4\bar{f}_1\bar{f}_3.$$

Let $L_i(j)$ and $R_i(j)$ denote the jth term of L_i and R_i, respectively. We can further define

$$T_1 = -R_2(1) + L_1(9), \quad T_2 = -R_2(4) + L_2(5),$$
$$T_3 = -R_2(2) + L_1(6), \quad T_4 = -R_2(3) + L_1(8),$$
$$T_5 = -R_1(1) + L_1(5), \quad T_6 = -R_1(4) + L_2(9),$$
$$T_7 = -R_1(2) + L_2(6), \quad T_8 = -R_1(3) + L_2(8).$$

Then, we want to prove

$$L_1 + L_2 - R_1 - R_2 = \sum_i T_i + \sum_{i=1,2,3,4,7} (L_1(i) + L_2(i)) > 0.$$

We now combine and simplify the following terms

$$T_1 + T_5 = (\Pi f_i)(f_2 + f_4 - 1)(f_1 + f_3 - 1),$$
$$T_2 + T_6 = (\Pi \bar{f}_i)(f_1 + f_3 - 1)(f_2 + f_4 - 1),$$
$$T_3 + T_4 + T_7 + T_8 = (f_1f_2\bar{f}_3\bar{f}_4 + f_3f_4\bar{f}_1\bar{f}_2)(f_1 + f_3 - 1)(f_2 + f_4 - 1).$$

We can then show

$$L_1 + L_2 - R_1 - R_2 = -(f_1 - f_3)(f_2 - f_4)(f_1f_2 + \bar{f}_1\bar{f}_2)(f_3f_4 + \bar{f}_3\bar{f}_4) + (f_1f_2\bar{f}_3\bar{f}_4 + f_3f_4\bar{f}_1\bar{f}_2)(f_1f_2 + \bar{f}_1\bar{f}_2 + f_3f_4 + \bar{f}_3\bar{f}_4 - 1).$$

Replacing all f_i with $g_i = f_i - \frac{1}{2}$, we have

$$L_1 + L_2 - R_1 - R_2$$
$$= -(g_1 - g_3)(g_2 - g_4)(2g_1g_2 + \frac{1}{2})(2g_3g_4 + \frac{1}{2}) + [-\frac{1}{2}(g_1 + g_2)(g_3 + g_4) + 2(g_1g_2 + \frac{1}{4})(g_3g_4 + \frac{1}{4})](2g_1g_2 + 2g_3g_4)$$
$$= 4g_2g_3(\frac{1}{4} - g_1^2)(\frac{1}{4} - g_4^2) + 4g_1g_4(\frac{1}{4} - g_2^2)(\frac{1}{4} - g_3^2).$$

Since $g_i \in (0, 0.5)$ (due to $f_i \in (0.5, 1)$), $\frac{1}{4} - g_i^2 > 0$. Therefore, $L_1 + L_2 - R_1 - R_2 > 0$, i.e., Equ. (43) holds.

Cost Part of Lemma 2

We need to prove the cost is reduced or kept the same after the transformation (in Fig. 1), i.e.,

$$\frac{\frac{c_1+c_3}{\Pr\{P_p(f_1,f_3)\}} + \frac{c_2+c_4}{\Pr\{P_p(f_2,f_4)\}}}{\Pr\{S_p\}} < \frac{\frac{c_1+c_2}{\Pr\{S_p\}} + \frac{c_3+c_4}{\Pr\{S_p\}}}{\Pr\{P_p(S_p(f_1,f_2), S_p(f_3,f_4))\}}. \tag{44}$$

Bringing in s_{ij} and expanding the probabilities, we need

$$\frac{c_1 + c_3}{s_{13}} + \frac{c_2 + c_4}{s_{24}} < \frac{\sum c_i}{s_{12}s_{34} + (1 - s_{12})(1 - s_{34})}.$$

Replacing all f_i with g_i, we have

$$\frac{c_1 + c_3}{2g_1g_3 + 1/2} + \frac{c_2 + c_4}{2g_2g_4 + 1/2} < \frac{\sum c_i}{8\Pi(g_i) + 1/2}.$$

To prove the last inequality, we need

$$\frac{c_1 + c_3}{2g_1g_3 + 1/2} < \frac{c_1 + c_3}{8\Pi(g_i) + 1/2} \Leftrightarrow 8\Pi(g_i) + 1/2 < 2g_1g_3 + 1/2 \Leftrightarrow g_2g_4 < 1/4,$$

and

$$\frac{c_2 + c_4}{2g_2g_4 + 1/2} < \frac{c_2 + c_4}{8\Pi(g_i) + 1/2} \Leftrightarrow g_1g_3 < 1/4.$$

Again since $g_i \in (0, 0.5)$ (due to $f_i \in (0.5, 1)$), both $g_1g_3 < 1/4$ and $g_2g_4 < 1/4$ hold. Therefore, Equ. (44) holds.

Proof of Theorem 3

We prove that the swapping node in TREE cannot be more efficient than its two children at the same time. We prove it by contradiction: assuming that both children of a swapping node are less efficient (so the swapping node is selected to purify), we can derive infeasible conditions. That is, at least one child is more efficient so that child (or its descendant) is selected to purify. To elaborate the proof, we first introduce more detailed notations. First, instead of solely using $\delta(A)$ to denote the efficiency of purifying the node A, we need to further point out the target node. By default, it is usually the root R. Now, we use $\delta(\frac{R}{A})$, defined as

$$\delta(\frac{R}{A}) = \frac{\Delta_f(\frac{R}{A})}{\Delta_c(\frac{R}{A})} = \frac{\Delta_f^P(A)G_f^P(\frac{R}{A})}{\Delta_c^P(A)G_c^P(\frac{R}{A})},$$

where $G_f(\frac{R}{A})$ is the gradient of fidelity of node A with respect to the node R, and $\Delta_f(\frac{R}{A})$ is the fidelity change of node R after purifying node A. Similarly, given that node B is a child of node A,

$$\delta(\frac{R}{B}) = \frac{\Delta_f(\frac{R}{B})}{\Delta_c(\frac{R}{B})} = \frac{\Delta_f^P(B)G_f(\frac{R}{B})}{\Delta_c^P(B)G_c(\frac{R}{B})}.$$

By the chain rule, given A is a SWAP node,

$$G_f(\frac{R}{B}) = G_f(\frac{R}{A})G_f^S(\frac{A}{B}), \quad G_c(\frac{R}{B}) = G_c(\frac{R}{A})G_c^S(\frac{A}{B}).$$

To lead to the contradiction, assuming both B and C (two children of A) are less efficient. To make B less efficient than A, we need

$$\frac{\delta(\frac{R}{A})}{\delta(\frac{R}{B})} > 1,$$

$$\frac{\Delta_f^P(A)G_f(\frac{R}{A})}{\Delta_c^P(A)G_c(\frac{R}{A})} \cdot \frac{\Delta_c^P(B)G_c(\frac{R}{B})}{\Delta_f^P(B)G_f(\frac{R}{B})} > 1,$$

$$\frac{\Delta_f^P(A)}{\Delta_c^P(A)} \cdot \frac{\Delta_c^P(B)G_c^S(\frac{A}{B})}{\Delta_f^P(B)G_f^S(\frac{A}{B})} > 1.$$

By definition, we know that $G_c^S(\frac{A}{B}) = \Pr\{S_p^A\}$, $G_f^S(\frac{A}{B}) = 2f_C - 1$, $\Delta_f^P(A) = \frac{f_A^2}{f_A^2+(1-f_A)^2} - f_A$, $\Delta_c^P(A) = \frac{2c_A}{f_A^2+(1-f_A)^2} - c_A$, $\Delta_f^P(B) = \frac{f_B^2}{f_B^2+(1-f_B)^2} - f_B$, and $\Delta_c^P(B) = \frac{2c_B}{f_B^2+(1-f_B)^2} - c_B$. Therefore, we need

$$\Pr\{S_p^A\}(\frac{f_A^2}{f_A^2+(1-f_A)^2} - f_A)(\frac{2c_B}{f_B^2+(1-f_B)^2} - c_B) > (2f_C - 1)(\frac{2c_A}{f_A^2+(1-f_A)^2} - c_A)(\frac{f_B^2}{f_B^2+(1-f_B)^2} - f_B).$$

By multiplying $[f_A + (1-f_A)^2][f_B + (1-f_B)^2]$ at both sides and substituting f_i by $g_i + \frac{1}{2}$, we need

$$\Pr\{S_p^A\}[(g_A+\frac{1}{2})^2 - (g_A+\frac{1}{2})(2g_A^2+\frac{1}{2})][2c_B - c_B(2g_B^2+\frac{1}{2})] > (2g_C)[(g_B+\frac{1}{2})^2 - (g_B+\frac{1}{2})(2g_B^2+\frac{1}{2})][2c_A - c_A(2g_A^2+\frac{1}{2})],$$

i.e.,

$$\Pr\{S_p^A\}(\frac{1}{2} - 2g_A^2)(\frac{3}{2} - 2g_B^2)c_B > (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 2g_A^2)c_A.$$

Since $g_A = 2g_Bg_C$, we need

$$\Pr\{S_p^A\}(\frac{1}{2} - 8g_B^2g_C^2)(\frac{3}{2} - 2g_B^2) - (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 8g_B^2g_C^2)\frac{c_A}{c_B} > 0.$$

Replacing c_A by $\frac{c_B+c_C}{\Pr\{S_p^A\}}$ and multiplying $\Pr\{S_p^A\}$ at both sides, we have

$$\Pr\{S_p^A\}^2(\frac{1}{2} - 8g_B^2g_C^2)(\frac{3}{2} - 2g_B^2) - (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 8g_B^2g_C^2)\frac{c_B+c_C}{c_B} > 0.$$

We use z_B to denote the left hand side.

Similarly, we have z_C for node C as

$$z_C = \Pr\{S_p^A\}^2(\frac{1}{2} - 8g_B^2g_C^2)(\frac{3}{2} - 2g_C^2) - (\frac{1}{2} - 2g_C^2)(\frac{3}{2} - 8g_B^2g_C^2)\frac{c_B+c_C}{c_C}.$$

For z_B, it is obvious that at the left hand side, the first term is positive and the second one is negative. Since $\Pr\{S_p^A\} \in [\frac{1}{2}, 1]$, if $\Pr\{S_p^A\} = 1$ cannot satisfy the inequality, then any other value of $\Pr\{S_p^A\}$ within this range cannot satisfy the inequality as well. Here, it is safe to assume that $\Pr\{S_p^A\} = 1$ and latter we will see that the inequality still cannot be satisfied. Then from $z_B > 0$, we have

$$(\frac{1}{2} - 8g_B^2 g_C^2)(\frac{3}{2} - 2g_B^2)c_B > (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 8g_B^2 g_C^2)(c_B + c_C) \tag{45}$$

$$2g_B^2 c_B > 8g_B^2 g_C^2 c_B + (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 8g_B^2 g_C^2)c_C \tag{46}$$

$$2g_B^2 c_B(1 - 4g_C^2) > (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 8g_B^2 g_C^2)c_C \tag{47}$$

$$c_B > (\frac{1}{2} - 2g_B^2)(\frac{3}{2} - 8g_B^2 g_C^2)c_C \Big/ 2g_B^2(1 - 4g_C^2) \tag{48}$$

The last step holds due to $\forall g_C \in [0, \frac{1}{2}]$, then $(1 - 4g_C^2) > 0$, $2g_B^2(1 - 4g_C^2) > 0$. Eq. (48) shows the lower bound of c_B. Then apply the same steps to z_C (except the last one), we can obtain similar result to Eq. (47),

$$2g_C^2 c_C(1 - 4g_B^2) > (\frac{1}{2} - 2g_C^2)(\frac{3}{2} - 8g_B^2 g_C^2)c_B \tag{49}$$

Then we can plug the lower bound of c_B in Eq. (48) into Eq. (49) to eliminate c_B and c_C. The rationale is that if the lower bound of c_B cannot make the inequality stand, then it would not be true anyway (as both sides are positive).

$$2g_C^2 c_C(1 - 4g_B^2) > (\frac{1}{2} - 2g_C^2)(\frac{3}{2} - 8g_B^2 g_C^2)^2 \frac{(\frac{1}{2} - 2g_B^2)c_C}{2g_B^2(1 - 4g_C^2)}$$

$$2g_C^2(1 - 4g_B^2)2g_B^2(1 - 4g_C^2) > \frac{1}{4}(1 - 4g_C^2)(\frac{3}{2} - 8g_B^2 g_C^2)^2(1 - 4g_B^2)$$

$$16g_B^2 g_C^2 > (\frac{3}{2} - 8g_B^2 g_C^2)^2 \quad \text{i.e., } |4g_B g_C| > \left|(\frac{3}{2} - 8g_B^2 g_C^2)\right|$$

so either

$$4g_B g_C > \frac{3}{2} - 8g_B^2 g_C^2 \tag{50}$$

or

$$4g_B g_C < -(\frac{3}{2} - 8g_B^2 g_C^2). \tag{51}$$

But (51) is always false because the left hand side is positive and the right hand side is negative. Solving (50), we can get $g_B g_C > \frac{1}{4}$ or $g_B g_C < -\frac{3}{4}$, which is impossible. Therefore, even we have relaxed $\Pr\{S_p^A\}$ and c_B in the way favorable to the inequalities set, it still cannot be true. That means that at least one of $z_B < 0$ and $z_C < 0$ should be true, so at least one of the nodes B and C are more efficient than A.